Chapter 13
Propagators of the real Klein Gordon field
–
from my book:
Understanding Relativistic Quantum Field Theory

Hans de Vries

November 11, 2008

Contents

Chapter 13

Propagators of the real Klein Gordon field

13.1 Klein Gordon propagator in position space

In this section we will study the propagation of the real Klein Gordon field from a hypothetical point source ζ. In position space we can write.

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \frac{m^2c^2}{\hbar^2} \right)\psi \;\;=\;\; \zeta(t,\vec{r}) \tag{13.1}$$

Or much more compact, using $c = 1$ and $\hbar = 1$

$$(\Box + m^2)\psi \;\;=\;\; \zeta(x^\mu) \tag{13.2}$$

We define $\zeta(x^\mu) = \delta(x^\mu)$ as a point-source in both space and time. It's Fourier transform in momentum space a constant over momentum and energy. We can write.

$$\left(-p_o^2 + p_x^2 + p_y^2 + p_z^2 + m\ \right)\psi \;\;=\;\; \left(-q^2 + m^2\ \right)\psi \;\;=\;\; \zeta(p^\mu) \tag{13.3}$$

The propagator in momentum space can now be written as.

$$\psi \;\;=\;\; \frac{-1}{q^2 - m^2}\,\zeta \qquad\Longrightarrow\qquad \mathcal{D}_p \;\;=\;\; \frac{-1}{q^2 - m^2} \tag{13.4}$$

The propagator for the Klein Gordon equation in momentum space is $-1/(q^2 - m^2)$, where q^2 is the shorthand notation for $\frac{1}{c^2}E^2 - p_x^2 - p_y^2 - p_z^2$.

This expression has poles (infinities) in momentum space for certain frequencies and they need careful attention. In position space we do not get infinite values from a plane-wave source. Why? where do they come? Actually the propagation from these plane-waves does become infinite but only after infinite time.

The reason behind this is that plane waves stretch out from x is minus infinity to plus infinity. Contributions from farther and farther away regions keep coming in and, at the pole frequency, they all add up. The result is infinite after t is infinte. Any physical process doesn't continue until t is infinite nor is infinite in size.

We can split up the physical process of Klein Gordon propagator with the following series expansion:

$$\frac{-1}{q^2 - m^2} \;=\; -\bigg(\frac{1}{q^2} + \frac{m^2}{q^4} + \frac{m^4}{q^6} + \frac{m^6}{q^8} + ... \tag{13.5}$$

Which becomes the following operator in configuration space:

$$\mathcal{D}(t,x) \;=\; \Box^{-1} - m^2\,\Box^{-2} + m^4\,\Box^{-3} - m^6\Box^{-4} + ... \tag{13.6}$$

Where $\Box^{-1}$ is the inverse d'Alembertian. This operator spreads the wave function out on the lightcone as if it was a massless field. The second term then retransmits it, opposing the original effect, again purely on the light cone. The third term is the second retransmission, et-cetera, ad-infinitum.

All propagators in this series are on the lightcone. The wave function does spread within the light cone because of the retransmission, but it does never spread outside the light cone, with superluminal speed.

We will solve the series first for the 1+1d dimensional case. We may rotate the E,p plane and the t,x plane by 45 degrees, they have the same Fourier transform correspondence, and use new variables defined by.

$$p_1 \;=\; \frac{E-p}{\sqrt{2}},\quad p_2 \;=\; \frac{E+p}{\sqrt{2}},\qquad u \;=\; \frac{t-x}{\sqrt{2}},\quad w \;=\; \frac{t+x}{\sqrt{2}} \tag{13.7}$$

$$\frac{1}{q^2} \;=\; \frac{1}{E-p}\,\frac{1}{E+p} \;=\; \frac{1}{2}\,\frac{1}{p_1\,p_2} \tag{13.8}$$

Note that E here is $\frac{1}{c}E$ with c set to 1. A division by a variable in the momentum domain amounts to an integration in the position domain.[1] We can now express the terms of the series (13.34) as a series of repeated integrations of the delta function which originates from the definition of the propagator.

[1] Accurately, we should do a convolution with the sign function here while the integration is a convolution with the Heaviside step-function $\theta(x)$. It does produce the exact result for the forward light-cone. Strictly causal propagation will be studied in detail in the succeeding sections

$$(\Box + m^2)\mathcal{D}(t,x) \;=\; \delta(t,x) \tag{13.9}$$

Writing out the series we get:

$$\begin{aligned}
&\Box^{-1}\delta(t,x) = \textstyle\int\int\ \tfrac{1}{2}[\ \ \delta(t,x)\ ]\ dudw && = \ \tfrac{1}{2}\ \theta(u)\theta(w) \\
&\Box^{-2}\delta(t,x) = \textstyle\int\int\ \tfrac{1}{2}[\tfrac{1}{2}\ \theta(u)\theta(w)\ ]\ dudw && = \ \tfrac{1}{4}\ \theta(u)\theta(w)\ u\ w \\
&\Box^{-3}\delta(t,x) = \textstyle\int\int\ \tfrac{1}{2}[\tfrac{1}{4}\ \theta(u)\theta(w)\ u\ w\ ]\ dudw && = \ \tfrac{1}{8}\ \theta(u)\theta(w)\ \tfrac{1}{2}u^2\tfrac{1}{2}w^2 \\
&\Box^{-4}\delta(t,x) = \textstyle\int\int\ \tfrac{1}{2}[\tfrac{1}{8}\ \theta(u)\theta(w)\tfrac{1}{2}u^2\tfrac{1}{2}w^2\ ]\ dudw && = \ \tfrac{1}{16}\ \theta(u)\theta(w)\ \tfrac{1}{6}u^3\tfrac{1}{6}w^3 \\
&\ldots\ldots
\end{aligned} \tag{13.10}$$

We can express these terms in the more familiar variable $s^2 = t^2 - x^2$. (The t used here is $c\,t$ with c set to 1).

$$uw \;=\; (t^2 - x^2)/2 = \; s^2/2 \tag{13.11}$$

$$\theta(u)\theta(w) \;=\; \theta(\frac{t^2 - x^2}{2}) \;=\; \theta(s^2/2) \tag{13.12}$$

So we can write for the series:

$$\mathcal{D}(t,x) \;=\; \frac{1}{2}\ \theta(s^2/2)\left(\ 1 - \frac{m^2s^2}{4} + \frac{m^4s^4}{16\ 2^2} - \frac{m^6s^6}{64\ 6^2} + ...\right) \tag{13.13}$$

The series between the brackets becomes a Bessel J function of order zero. To see this we use the series expansion for the general Bessel J function.

$$J_\nu(z) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+\nu)!\ k!}\left(\frac{z}{2}\right)^{2k+\nu} \tag{13.14}$$

In our case we need the zero order Bessel J function:

$$J_0(z) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{z}{2}\right)^{2k} \tag{13.15}$$

The Bessel J_0 series is identical to the one in brackets in the expression for the propagator (13.13), so we use it to replace the series:

$$\mathcal{D}(t,x) \;=\; \frac{1}{2}\,\theta\left(s^2/2\right) \mathrm{J}_0(ms) \tag{13.16}$$

We can extend this to any dimensional space using the "inter-dimensional operator" The Bessel J function of order zero becomes one of first order in 3+1d space.

$$\mathcal{D}_d(t,r) \;=\; \frac{1}{\pi^a}\,\frac{\partial^a}{\partial(s^2)^a}\,\mathcal{D}_1(t,r), \qquad \left(a = \tfrac{d-1}{2}\right) \tag{13.17}$$

Where d is the number spatial dimensions (3 in this case). From the series expansion for Bessel functions (13.14) we can derive an expression to take derivatives of Bessel functions.

$$\frac{\partial\left(z^{-\nu}\ \mathrm{J}_\nu(z)\right)}{\partial z} \;=\; -z^{-\nu}\ \mathrm{J}_{\nu+1}(z) \tag{13.18}$$

Which we can use to apply the inter-dimensional operator (13.36). As the final result we now get for the position space propagator on the forward light-cone.

Real field Klein Gordon propagator in position space

$$\mathcal{D}(t,r) \;=\; \theta(t)\left(\ \frac{1}{4\pi}\,\delta\left(s^2/2\right)\ -\frac{1}{4\pi}\frac{m}{s}\,\theta\left(s^2/2\right)\ \mathrm{J}_1(ms)\ \right) \tag{13.19}$$

$$s^2 = t^2 - x^2$$

The inter-dimensional operator is proved in appendix .. In the massless limit this expression becomes the Green's function of the classical wave equation. The argument used in the delta function $(s^2/2)$ is the rigorous mathematical version. It can be replaced by one of several equivalent

versions with equal volume if the exact shape of the delta function is not important.

$$\int \frac{1}{4\pi}\,\delta\left(s^2/2\right) \;=\; \int \frac{1}{2\pi}\,\delta\left(s^2\right) \;=\; \int \frac{1}{4\pi r}\,\delta\left(t-|r|\right) \tag{13.20}$$

The rightmost one explicitly shows the $1/r$ potential field solution.

13.2 Real field Klein Gordon forward propagator

A strictly causal propagator should propagate only within the forward light-cone. In the previous section we showed that the standard Klein Gordon propagator $1/(E^2-p^2-m^2)$ propagates within the light-cone. However, the propagator is symmetrical in E and thus also symmetrical in time according to the symmetry-properties of the Fourier transform.

The propagator is *Even* in time, $\mathcal{D}(-t,r) = \mathcal{D}(t,r)$ We have to find the *Hilbert partner* of the momentum-space propagator which represents the *Odd* propagator $\mathcal{D}(-t,r) = -\mathcal{D}(t,r)$, so that we can obtain:

$\frac{1}{2}(Even + Odd) =$ Forward propagator.

$\frac{1}{2}(Even - Odd) =$ Backward propagator.

The antisymmetric Odd propagator is simply the symmetric propagator multiplied by the sign function. The derivation of the causal, forward in time, Klein Gordon propagator follows closely that of the forward photon propagator. We start with the propagator which is symmetric in the energy E. (and thus also symmetric in time)

$$\mathcal{D}^{+\triangledown}_{+\vartriangle}(E,p) \;=\; \frac{1}{E^2-p^2-m^2} \tag{13.21}$$

Which corresponds to a propagator in position space which is symmetric in time. The triangles in the superscript and subscript symbolize the forward and backward light-cone. We can separate this propagator in a positive and negative pole.

$$\frac{1}{E^2-p^2-m^2} \;=$$

$$\frac{1}{2\sqrt{p^2+m^2}}\left(\frac{1}{E-\sqrt{p^2+m^2}} - \frac{1}{E+\sqrt{p^2+m^2}}\right) \tag{13.22}$$

We see that both poles propagate both positive and negative energy plane-waves even though their peaks are in different halves. Methods which attempt to separate positive and negative frequencies by picking either one of the two poles are mathematically incorrect.

The anti-symmetric propagator is the Hilbert partner of the symmetric propagator. The two can be transformed into each other by a convolution with $i/(\pi E)$.

$$\mathcal{D}^{+\nabla}_{-\Delta}(E,p) = \left(\frac{i}{\pi E}\right) * \mathcal{D}^{+\nabla}_{+\Delta}(E,p) \tag{13.23}$$

$$\mathcal{D}^{+\nabla}_{+\Delta}(E,p) = \left(\frac{i}{\pi E}\right) * \mathcal{D}^{+\nabla}_{-\Delta}(E,p) \tag{13.24}$$

The square root is just a constant in the above operations. The correct correspondence with the massless photon propagator is given by the replacement.

$$p \quad \Leftrightarrow \quad +\sqrt{p^2+m^2} = \omega_p \tag{13.25}$$

We should keep in mind that the p we are using corresponds to the frequency domain of r. We could have given it a subscript like in p_r. If we make the above replacement in equation (**??**) for the anti-symmetric photon propagator then we obtain the anti-symmetric (odd) Klein Gordon propagator.

$$\mathcal{D}^{+\nabla}_{-\Delta}(E,p) = -\frac{i\pi}{\omega_p}\Big(\delta\left(E-\omega_p\right) - \delta\left(E+\omega_p\right)\Big) \tag{13.26}$$

This propagator propagates only on the mass shell. Once we have the symmetric (even) and anti-symmetric (odd) propagator we can combine them to get the Forward and backward in time propagators.

$$\text{Forward propagator} = \tfrac{1}{2}(\text{ Even propagator} + \text{Odd propagator })$$

$$\mathcal{D}^{\triangledown}(E,p) \quad = \quad \frac{1}{2}\left(\; \mathcal{D}^{+\triangledown}_{+\vartriangle}(E,p) \; + \; \mathcal{D}^{+\triangledown}_{-\vartriangle}(E,p) \; \right) \tag{13.27}$$

Backward propagator = $\frac{1}{2}$(Even propagator - Odd propagator)

$$\mathcal{D}_{\vartriangle}(E,p) \quad = \quad \frac{1}{2}\left(\; \mathcal{D}^{+\triangledown}_{+\vartriangle}(E,p) \; - \; \mathcal{D}^{+\triangledown}_{-\vartriangle}(E,p) \; \right) \tag{13.28}$$

So we obtain the:

Causal, forward in time, real field Klein Gordon propagator

$$\mathcal{D}^{\triangledown}(E,p) \quad = \quad \frac{1}{E^2-p^2-m^2} \quad - \quad \frac{i\pi}{2\omega_p}\Big(\delta\left(E-\omega_p\right)-\delta\left(E+\omega_p\right)\Big) \tag{13.29}$$

We see that the modification to make the Klein Gordon propagator causal only changes the behavior at the poles, as was the case with the forward photon propagator. We reorganize the forward propagator on a pole-by-pole base to study the behavior at the poles:

$$\mathcal{D}^{\triangledown}(E,p) \; = \; \frac{1}{2\omega_p}\left(\frac{1}{E-\omega_p} \; - \; i\pi\;\delta\left(E-\omega_p\right) \;\; - \;\; \frac{1}{E+\omega_p} \; + \; i\pi\;\delta\left(E+\omega_p\right) \right) \tag{13.30}$$

The on-the-mass-shell propagation would be ill defined without the addition of the delta-functions which have a magnitude infinitely much higher as that of the reciprocal poles alone. See the discussion with Rayleigh's theorem (**??**)

Somewhat symbolically we can write for the amplitude of the propagation if we change the frequency slowly from one side of the pole to the other side of the pole.

$$-\infty \quad \rightarrow \quad -\,i\infty^2 \quad \rightarrow \quad +\,\infty \tag{13.31}$$

The middle value -$i\infty^2$ symbolizes the contribution of the delta function. A real particle with a finite lifetime has a frequency spectrum which is

not infinitely sharp. The spectrum extends at both sides of the pole. There would be destructive interference since the amplitude at both side is opposite in sign.

Minimal changes in frequency would move the center frequency to either side of the poles and the destructive interference would disappear. The addition of the dominating delta function in the middle of the pole makes the on-shell propagation well defined.

Finally, we also obtain the backward propagator. We do so by simply subtract the odd propagator instead of adding it.

Backward in time, real field Klein Gordon propagator

$$\mathcal{D}_{\vartriangle}(E,p) \quad = \quad \frac{1}{E^2-p^2-m^2} \quad + \quad \frac{i\pi}{2\omega_p}\Big(\delta\left(E-\omega_p\right)-\delta\left(E+\omega_p\right)\Big) \tag{13.32}$$

13.3 The KG propagator expanded in the mass term

We expanded the real Klein Gordon propagator in momentum space into the mass term m^2 when we calculated the Klein Gordon propagator in position space.

$$\frac{1}{E^2-p^2-m^2} \;=\; \frac{1}{E^2-p^2} \;+\; \frac{m^2}{(E^2-p^2)^2} \;+\; \frac{m^4}{(E^2-p^2)^3} \;+\; .... \tag{13.33}$$

In this section we will explore this expansion a bit more. We did see that the series can be written in configuration space as.

$$\mathcal{D}(t,x) \;=\; \Box^{-1} \;-\; m^2\,\Box^{-2} \;+\; m^4\,\Box^{-3} \;-\; m^6\,\Box^{-4} \;+\; .... \tag{13.34}$$

Where $\Box^{-1}$ is the inverse d'Alembertian operator. The first term of these series corresponds to the massless propagator. The subsequent terms can

be interpreted as the first, second and third retransmission based on the m^2 term.

We can express this series in the inverse d'Alembertian operator as a series of Green's functions expressed in the parameter s^2, $(= t^2 - r^2)$ We do so by applying the inter-dimensional operator (13.36) on the 1d series (13.13) to obtain the 3d series. The result is:

$$\mathcal{D}(t,x) \;=\; \frac{\delta(s^2/2)}{4\pi} \;-$$
$$\frac{\theta(s^2/2)}{2\pi}\left(\frac{m^2}{2^2} \;-\; \frac{m^4 s^2}{2^4 1!\; 2!} \;+\; \frac{m^6 s^4}{2^6 2!\; 3!} \;-\; \frac{m^8 s^6}{2^8 3!\; 4!} \;+\; ....\right) \qquad (13.35)$$

The first term is the Green's function of the massless particle, equal to that of the photon. The remaining series can be written as a Bessel J function of first order, see (13.19)

Another intresting observation is that the momentum space series (13.33) represents the 1d, 3d, 5d, ... Fourier transforms of the 1d, 3d, 5d solid light cones. A solid light-cone has a constant value everywhere inside the light-cone and a zero value outside the light-cone.

We can see why by repeatedly applying the inter-dimensional operator to the Green function series in s^2 to increase the number of spatial dimensions by 2 each step. This comes down simply to a differentiation in s^2 per step:

$$\mathcal{D}_{d+2}(t,r) \;=\; \frac{1}{\pi}\,\frac{\partial}{\partial(s^2)}\;\mathcal{D}_d(t,r) \qquad (13.36)$$

Any term in the form of: $\theta(s^2/2)\; s^{2n}$ will become constant within the light-cone $\theta(s^2/2)$ after n steps. This further means that the series expressed in the intermediate time/momentum domain represents the Fourier transforms of the 1d, 3d, 5d ... solid spheres.

From (**??**) and (**??**) and substitution $p \Rightarrow \mathrm{sgn}(p)\,\sqrt{p^2+m^2}$ we can derive the symmetric in time and Klein Gordon propagator in the t,p domain.

$$\mathcal{D}^{+\nabla}_{+\Delta}(t,p) \;=\; \frac{\sin\left(\sqrt{p^2+m^2}\;\;|t|\right)}{\sqrt{p^2+m^2}} \qquad (13.37)$$

As well as the anti-symmetric in time Klein Gordon propagators in the t,p domain.

$$\mathcal{D}^{+\nabla}_{-\Delta}(t,p) \;=\; \frac{\sin\left(\sqrt{p^2+m^2}\;\;t\right)}{\sqrt{p^2+m^2}} \tag{13.38}$$

We can expand these expression as a series in t.

$$\frac{\sin\left(\sqrt{p^2+m^2}\;t\right)}{\sqrt{p^2+m^2}} \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\;\left(p^2+m^2\right)^n\;\;t^{2n+1} \tag{13.39}$$

Which does not contain any square roots anymore. The next step is to expand these in the mass term m^2 which gives us.

$$\begin{aligned}
\mathcal{D}(t,p) \;&=\; \tfrac{1}{p}\Big[\;\; pt \;\; -\tfrac{1}{3!}t^3p^3 \;\; +\tfrac{1}{5!}t^5p^5 \;\; -\tfrac{1}{7!}t^7p^7 \;\; +\tfrac{1}{9!}t^9p^9 ... \\
&+\; \tfrac{m^2}{p^3}\Big[\;\; -\tfrac{1}{3!}t^3p^3 \;\; +\tfrac{2}{5!}t^5p^5 \;\; -\tfrac{3}{7!}t^7p^7 \;\; +\tfrac{4}{9!}t^9p^9 ... \\
&+\; \tfrac{m^4}{p^5}\Big[\;\; +\tfrac{1}{5!}t^5p^5 \;\; -\tfrac{3}{7!}t^7p^7 \;\; +\tfrac{6}{9!}t^9p^9 ... \\
&+\; \tfrac{m^6}{p^9}\Big[\;\; -\tfrac{1}{7!}t^7p^7 \;\; +\tfrac{4}{9!}t^9p^9 ... \\
&+\; .....
\end{aligned} \tag{13.40}$$

Where the denominators correspond with the binomial triangle. Each of these series can be written as a finite length series in sine and cosine terms.

$$\begin{aligned}
\mathcal{D}(t,p) \;=\; & \left[\ \tfrac{1}{pt}\ \sin(pt)\ \right] t \\
- \; & \tfrac{1}{2}m^2 \left[\ \tfrac{1}{(pt)^2}\sin(pt) - \tfrac{1}{pt}\cos(pt)\ \right] t^2p^{-1} \\
+ \; & \tfrac{1}{8}m^4 \left[\ \tfrac{3}{(pt)^3}\sin(pt) - \tfrac{3}{(pt)^2}\cos(pt) - \tfrac{1}{pt}\sin(pt)\ \right] t^3p^{-2} \\
- \; & \ldots\ldots
\end{aligned} \tag{13.41}$$

The first term is the 1d Fourier transform of the square-pulse function, which can be seen as the 1d solid sphere. The second is the 3d Fourier transform of the solid sphere. The next series is the 5d Fourier transform of the 5d solid sphere. et-cetera.

The terms between the square brackets are the *spherical Bessel functions* of the first kind which are generated by repeatedly differentiating the sync function.

$$j_n(x) \;=\; (-x)^n \left(\frac{1}{x}\,\frac{d}{dx}\right)^n \frac{\sin x}{x} \tag{13.42}$$

The first few spherical Bessel functions generated in this way are.

$$j_0(x) \;=\; \frac{\sin x}{x} \tag{13.43}$$

$$j_1(x) \;=\; \frac{\sin x}{x^2} - \frac{\cos x}{x} \tag{13.44}$$

$$j_2(x) \;=\; \frac{3\sin x}{x^3} - \frac{3\cos x}{x^2} - \frac{\sin x}{x} \tag{13.45}$$

The generator of the spherical Bessel functions is thus related to our inter dimensional operator from which we can derive the radial Fourier transform of a function in any d-dimension from the simplest, one dimensional case.

$$\mathcal{D}_d(t,r) \;=\; \frac{1}{\pi^a}\,\frac{\partial^{\,a}}{\partial(s^2)^a}\,\mathcal{D}_1, \qquad \left(a = \tfrac{d-1}{2}\right) \tag{13.46}$$

13.4 The square root Hamiltonian

An operator of interest is the square root Hamiltonian. A direct linearization of the expression.

$$-\hbar^2 \frac{\partial}{\partial t}\,\psi \;\; = \;\; \tilde{H}^2\,\psi \;\; = \;\; \left(\tilde{\mathbf{p}}^2 c^2 + m^2 c^4\right)\psi \tag{13.47}$$

Where the tildes on the $\tilde{H}$ and $\tilde{p}$ defines them as operator. We can use her the series.

$$\sqrt{1+\tilde{p}^2} \;\; = \tag{13.48}$$

$$1 + \tfrac{1}{2}\tilde{p}^2 - \tfrac{1}{8}\tilde{p}^4 + \tfrac{1}{16}\tilde{p}^6 - \tfrac{5}{128}\tilde{p}^8 + \tfrac{7}{256}\tilde{p}^{10} - \tfrac{21}{1024}\tilde{p}^{12} + \tfrac{33}{2048}\tilde{p}^{14} - \tfrac{429}{32768}\tilde{p}^{16} + ...$$

This series is found to be represented by a finite expression and we can express $\tilde{H}$ in the form of the following expression.

$$i\hbar\frac{\partial}{\partial t}\,\psi \;\; = \;\; \tilde{H}\,\psi \;\; = \;\; \sqrt{\tilde{\mathbf{p}}^2 c^2 + m^2 c^4}\;\psi \;\; =$$

$$= \;\; \pm\; mc^2 \left\{ 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\;(2n-2)!}{n!(n-1)!\;2^{2n-1}} \left(\frac{\tilde{\mathbf{p}}^2}{m^2c^2}\right)^n \right\}\psi \tag{13.49}$$

Or written out as a differential operator:

$$i\hbar\frac{\partial}{\partial t}\,\psi \;\; = \;\; \tilde{H}\,\psi \;\; = \tag{13.50}$$

$$\pm\; mc^2 \left\{ 1 - \sum_{n=1}^{\infty} \frac{(2n-2)!}{n!(n-1)!\;2^{2n-1}} \left(\frac{\hbar^2}{m^2c^2}\right)^n \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^n \right\}\psi$$

Finally, including the EM interactions we get:

$$i\hbar\frac{\partial}{\partial t}\,\psi \;\; = \;\; \tilde{H}\,\psi \;\; = eV\psi \tag{13.51}$$

$$\pm\; mc^2 \left\{ 1 - \sum_{n=1}^{\infty} \frac{(2n-2)!}{n!(n-1)!\;2^{2n-1}} \left(\frac{1}{m^2c^4} \;\sum_{i=1}^{3} \left(c\hbar\frac{\partial}{\partial x_i} - ieA_{x_i}\right)^2 \right)^n \right\}\psi$$

In the chapter on the propagators of the complex Klein Gordon fields we will find a way to calculate this $\tilde{H}$ via different way and find that the application of $\tilde{H}$ is represented by a convolution with a Green's function containing Bessel functions.